The 'Ups' and 'Downs' of Subtraction: Young Children's Additive and Subtractive Mental Strategies for Solutions of Subtraction Word Problems and Algorithmic Exercises

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This paper reports on a longitudinal study across Years 2 and 3 of 104 children's performance and strategy use for two-digit separation and missing-addend word problems and vertical and horizontal algorithmic exercises with and without regrouping. The study showed that while Year 2 children predominantly used a subtractive *removing* strategy for separation problems and an additive *building-on* strategy for missing-addend problems, Year 4 children were more mixed in their strategy use. It also showed that children used subtractive strategies predominantly for vertical and horizontal algorithmic exercises.

A subtraction example can be solved *subtractively* by removing the minuend from the subtrahend (with materials, by counting down or by subtracting components) or *additively* by building from the minuend to the subtrahend (with materials, counting up or by adding components). Research has indicated that such subtractive and additive strategies or procedures are used spontaneously by young children for word problems (e.g., Carpenter & Moser, 1984), by young children for number facts (e.g., Steinberg, 1985), and by older Years 8 to 12 children for algorithmic exercises (e.g., Perry & Stacey, 1994); and can be taught to children (e.g. Fuson, 1986a, 1986b; Fuson & Willis, 1988; Thornton, 1990). Subtraction word problems can be presented with a subtractive or separation semantic structure (e.g., *There were 6 apples and 2 were removed. How many apples are left?*) and an additive or missing-addend semantic structure (e.g., *There were 2 apples. More apples were added to the bowl. There are now 6 apples in the box. How many were added?*) (Carpenter & Moser, 1984).

Research on relationships between children's strategy use and semantic structure for subtraction problems has been inconclusive and contradictory. Some studies have reported a direct relationship. Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) found that kindergarten children used subtractive strategies for separation problems and additive strategies for missing-addend problems. Similarly, De Corte and Verschaffel (1987) found that Years 1 to 3 children used procedures which modelled the semantic structure of the word problems. Other studies have indicated that this relationship only holds for young children. Carpenter (1986) reported that older children are not limited to procedures that directly match problem structure. In a 3 year longitudinal study of addition and subtraction strategies, Carpenter and Moser (1984) reported that Years 1 to 3 children solved missing-addend problems additively, but only initially solved separation problems subtractively. Later, the children tended to solve separation problems additively. Carpenter and Moser (1983) postulated that additive strategies are used earlier and with higher frequency than subtractive strategies, and that some children may never use subtractive strategies. However, it should be noted that the number combinations used in this study favoured the additive *count up* strategy.

This finding that young children prefer additive strategies for subtraction problems has been supported by yet other studies. Secada (1982) and Steffe, Spikes, and Hirsten (1976) reported that Year 1 children preferred additive to subtractive strategies for both separation and missing-addend problems. Fuson (1986a, 1986b) and Fuson and Willis (1988) reported that Year 2 children also preferred additive over subtractive strategies for separation and missing-addend problems, and that these children learnt multidigit subtraction earlier than is normal for that Year level by employing an additive method. Boulton-Lewis (1993) argued that children favoured additive strategies because they involved less processing load than the subtractive strategies.

This preference for additive strategies has not been found in studies of basic facts and algorithmic exercises. Steinberg (1985) reported a higher incidence of subtractive strategies in a study of Year 2 children's spontaneous derived facts strategies. Heirdsfield (1996) found that Year 4 children did not appear to prefer additive strategies for subtraction number facts. She reported that the dominant strategy was *immediate fact recall* (50.4%), with additive strategies 16.5% and subtractive strategies 19.5%. She found that subtractive strategies were predominantly used when the example proved too difficult to be solved using *immediate fact recall.* In a longitudinal study of 82 Years 1 to 3 children's understanding of multidigit addition and subtraction operations, Carpenter, Franke, Jacobs and Fennema (1996) found missing-addend problems more difficult for children to solve than separation problems.

This paper reports on a 2 year ARC funded longitudinal study of Years 2 to 4 children's mental strategies for addition and subtraction. Two and three digit separation and missing-addend word problems, and two and three digit algorithmic exercises were presented to children in Years 2 and 3, and beginning of Year 4 (some results of this study are presented in Cooper, Heirdsfield, & Irons, 1995a, 1995b, in press). Mental strategies refers to those strategies children employ to perform "arithmetic calculations without the aid of external devices" (Sowder, 1988, p. 182). As mental strategies were not specifically taught to the children in the study, it was hoped to elicit invented strategies. In order to increase the likelihood of children's employing invented strategies, no materials (concrete and pen and paper) were available. Carpenter and Moser (1984) have reported that children often use less advanced and less efficient strategies than they are capable of if materials are available. The paper compares the children's choices of additive and subtractive mental strategies for 2 digit separation and missing-addend word problems, and describes the children's strategy choice for subtraction algorithmic exercises. Efficiency of strategy use over the two years is also discussed.

The Study

Subjects

The subjects were 104 children of varying mathematical abilities (one third each of above average, average, and below average) in 6 Brisbane primary schools (3 State and 3 Catholic) representing a variety of social backgrounds. Although the teachers selected the children, their mathematical abilities were not revealed to the researchers to avoid bias in the interviews. The children participated in the study from the beginning of year 2 (1991) to the beginning of year 4 (1993).

Interview procedures

The interview method was Piaget's revised clinical interview technique (Ginsburg, Kossan, Schwartz, & Swanson, 1983). The questions (for the purposes of this paper) consisted of 2 digit separation and missing-addend subtraction word problems relating to money, and 2 digit algorithmic exercises (vertical and horizontal), presented in visual and oral form (the interviewer read the questions). The problem types are summarised in Table 1 (based on Carpenter & Moser, 1984).

Table 1
Subtraction word problems

Problem type	Example
separation (subtractive)	If John had 82 cents and spent 54 cents on bananas, how much money does he have left?
missing-addend (additive)	If Nancy had 47 cents and the chocolate costs 75 cents, how much extra money does she need?

Further, examples with and without regrouping were included for both word problems and algorithmic exercises in most interviews. The types of numbers involved in all subtraction questions are summarised in Table 2. As much as possible, numbers for the questions were chosen to elicit a range of spontaneous strategies, in particular, wholistic.

Table 2
Subtraction question types

Question type	Example (of numbers)
Word problems	
separation, 2 digits, no regrouping	48 - 25
separation, 2 digits, regrouping	95 - 47
missing-addend, 2 digits, no regrouping	86 - 30
missing-addend, 2 digits, regrouping	92 - 57
Algorithmic exercises	
horizontal algorithm, 2 digits, no regrouping	68-23
horizontal algorithm, 2 digits, regrouping	55-27
vertical algorithm, 2 digits, no regrouping	89 - 37
vertical algorithm, 2 digits, regrouping	45 - 26

The structure of the interview was to present questions for each problem type in increasing order of difficulty, until the children's responses indicated they were experiencing difficulties. Further, as children progressed in their ability to solve more difficult questions, easier questions (for instance, 2 digit subtraction without regrouping) were no longer presented to them. Thus, by interview 4, fewer children were presented with these easy questions, in order to maintain a maximum length of the interview. Further, vertical algorithms for examples involving no regrouping and horizontal algorithms (both regrouping and no regrouping) were no longer presented.

The children were interviewed 6 times: beginning, middle, and end of year 2 (interviews 1, 2, and 3), beginning and end of year 3 (interviews 4 and 5), and beginning of year 4 (interview 6). However, interview 4 contained no missing-addend problems and is not reported here.

The children were withdrawn from the classroom and interviewed on a one to one basis in a separate room. The interviews, whose length was kept to a maximum of 30 minutes, were videotaped. The questions were presented visually, as the researcher read each question and asked the children to explain their solution strategy. No materials (concrete or pen and paper) were provided for the children as aids in calculations. However, they were permitted to use their fingers.

Analysis

The videotapes were transcribed into protocols and behaviours analysed for strategy choice. The coded responses for each interview were tabulated and analysed for each question type. The strategies were categorised in terms of whether both numbers were split into place values, giving rise to strategies called *right to left separated* and *left to right separated* (depending on whether the ones or tens were computed first), *aggregation* (depending on whether only one number was split into place values), or *wholistic* (numbers were treated as a whole and not split into place values (reported in Cooper, Heirdsfield, & Irons, 1995a, 1995b, in press). For this paper, analysis was aimed at identifying additive and subtractive strategies for each question type over the 5 interviews. Examples of *additive*, *subtractive*, and *combined* strategies are presented in Table 3. *Combined* refers to both additive and subtractive processes being used in the same example or example type, for instance, using an additive process to calculate the tens, and subtracting to calculate the ones.

Table 3
Mental computation strategies

Strategies	Example (for 95-47)
R→L Separated	
subtractive	15-7=8, 80-40=40, 48
additive	7+8=15, 50+40=90, 48
combined	7+8=15, 80-40=40, 48
L→R Separated	
subtractive	90-40=50, 50-10=40, 15-7=8, 48
additive	40+50=90, 7+8=15, one less 10, 48
combined	90-40=50, 50-10=40, 7+8=15, 48
Aggregation	
subtractive	95-40=55, 55-7=48; or 95-7=88, 88-40=48
additive	47+40=87, 87+8=95, 48; or 47+8=55, 55+40=95, 48
combined	47+40=87, 95-7=88, 40+8=48
Wholistic	
subtractive	95-50+3=48
additive	45+50=95, 50-2=48

Results

The results for each question type for each interview are presented in Table 4. The data reported include the percentage of the population who were able to attempt the question type at each interview, the percentage of correct responses for those who were able to attempt the question type; then considering again the children who attempted the question type, the percentage who employed additive, subtractive or combined strategies, and finally, the percentage of those using each type of strategy who were correct. Thus, efficiency of strategies may be compared, but small numbers employing strategies must be kept in mind when comparing some choices.

General trends for question types indicate that an increasing percentage of students were able to attempt questions over the interviews, and accuracy levels tended to improve. Further, more children were able to attempt separation word problems than missing-addend problems, although accuracy levels were about the same (taking into account the number of students attempting each question type). For examples that did not require regrouping, more students were able to attempt missing-addend problems than algorithmic exercises; however, the trend was reversed for regrouping examples where more children attempted algorithmic exercises than missing-addend problems. Further, there is little difference in accuracy levels for missing-addend problems and algorithmic exercises, although it may be hazardous to draw such conclusions with small numbers .

Subtractive strategies were predominantly employed for separation word problems and algorithmic exercises, and additive strategies were predominantly used for missing-addend word problems. Overall, there is little difference in the accuracy levels of strategy choice, that is, whether additive, subtractive or a combination strategies were chosen had little effect on accuracy. However, very small numbers for some strategy use for some question types made it difficult to compare accuracy levels. Although separation word problems and algorithmic exercises were predominantly solved using subtractive strategies, a higher percentage employed subtractive for algorithmic exercises. In the first few interviews, most students solved word problems using the strategy that reflected the semantic structure of the problem. However, by the beginning of year 4, a greater variety of strategies was evident, although most solution strategies still reflected the question structure. Thus, as the children progressed through their schooling, they were not necessarily limited to procedures that directly matched the semantic structure of the problem. Responses to algorithmic exercises showed less variation. It appears that students consistently considered algorithmic exercises as subtractive.

Table 4
Children's performance and strategy use for subtraction problems and exercises

Measure	Strategies	Interviews 1 (*n*=104)	2 (*n*=104)	3 (*n*=90)	5 (*n*=104)	6 (*n*=104)
A. Separation, 2dig. probs no rgp						
% attempting question (of *n*)		24.3	62.5	72.2	73.1	64.4
% correct (of those attempting ques.)		38.6	52.3	60	77.6	86.6
% who used denoted strategy	additive	13.6	7.7	10.8	19.7	23.9
(of those attempting question)	subtractive	81.8	82.4	80	61.8	64.2
	combined	4.5	10.8	9.2	11.8	9
% correct (of those using	additive	33.3	40	85.7	80	70.6
denoted strategy)	subtractive	41.7	50.9	57.7	80.9	93
	combined	0	71.4	50	88.9	100
B. Miss-add, 2dig. probs no rgp						
% attempting question (of *n*)		34.6	51	55.6	57.7	44.2
% correct (of those attempting ques.)		38.9	60.4	52	81.7	71.7
% who used denoted strategy	additive	86.1	92.5	90	83.3	62.2
(of those attempting question)	subtractive	13.9	7.5	10	15	32.6
	combined	0	0	0	1.7	2.2
% correct (of those using	additive	38.7	59.2	46.7	82	63.3
denoted strategy)	subtractive	40	75	100	88.9	86.7
	combined	NA	NA	NA	0	100
C. Alg Ex, 2dig. no rgp vertical						
% attempting question (of *n*)		25	34.6	26.7	NA	NA
% correct (of those attempting ques.)		26.9	47.2	45.8		
% who used denoted strategy	additive	0	2.8	8.3		
(of those attempting question)	subtractive	100	97.2	91.7		
	combined	0	0	0		
% correct (of those using	additive	NA	0	0		
denoted strategy)	subtractive	26.9	48.6	50		
	combined	NA	NA	NA		
D. Alg Ex, 2dig. no rgp hor						
% attempting question (of *n*)		19.2	38.5	44.4	NA	NA
% correct (of those attempting ques.)		15	42.5	50		
% who used denoted strategy	additive	0	0	5		
(of those attempting question)	subtractive	100	100	92.5		
	combined	0	0	2.5		
% correct (of those using	additive	NA	NA	0		
denoted strategy)	subtractive	15	42.5	54.1		
	combined	NA	NA	0		
E. Separation, 2dig. probs rgp						
% attempting question (of *n*)		4.8	14.4	18.9	68.3	77.9
% correct (of those attempting ques.)		40	26.7	35.3	28.2	51.9
% who used denoted strategy	additive	0	20	17.6	9.9	3.7
(of those attempting question)	subtractive	100	60	82.4	70.4	75.3
	combined	0	20	0	19.7	21
% correct (of those using	additive	NA	33.3	66.7	57.1	0
denoted strategy)	subtractive	40	33.3	28.6	26	45.9
	combined	NA	0	NA	21.4	82.4
F. Miss -add, 2dig. probs rgp						
% attempting question (of *n*)		1.9	3.8	5.6	21.2	37.5
% correct (of those attempting ques.)		50	25	0	50	61.5
% who used denoted strategy	additive	100	75	100	72.7	59
(of those attempting question)	subtractive	0	0	0	18.2	35.9
	combined	0	25	0	9.1	5.1
% correct (of those using	additive	50	33.3	0	62.5	47.8
denoted strategy)	subtractive	NA	NA	NA	25	78.6
	combined	NA	0	NA	0	100

Table 4 (cont.)
Children's performance and strategy use for subtraction problems and exercises

Measure	Strategies	Interviews 1 (*n*=104)	2 (*n*=104)	3 (*n*=90)	5 (*n*=104)	6 (*n*=104)
G. Alg Ex, 2dig. rgp vertical						
% attempting question (of *n*)		7.7	9.6	10	63.5	62.5
% correct (of those attempting ques.)		50	40	44.4	25.8	46.2
% who used denoted strategy	additive	0	0	0	1.5	4.6
(of those attempting question)	subtractive	100	90	88.9	98.5	95.4
	combined	0	10	11.1	0	0
% correct (of those using	additive	NA	NA	NA	0	0
denoted strategy)	subtractive	50	44.4	50	26.2	48.4
	combined	NA	0	0	NA	NA
H. Alg Ex, 2dig. rgp hor						
% attempting question (of *n*)		2.9	4.8	13.5	NA	NA
% correct (of those attempting ques.)		0	60	28.6		
% who used denoted strategy	additive	0	0	0		
(of those attempting question)	subtractive	100	100	100		
	combined	0	0	0		
% correct (of those using	additive	NA	NA	NA		
denoted strategy)	subtractive	0	60	28.6		
	combined	NA	NA	NA		

An increasing percentage of children attempted separation, 2 digit without regrouping word problems from interviews 1 through 5 (Table 4, A). The decrease in interview 6 resulted from a growing number of students being able to attempt the more difficult regrouping problems, and thus, not being presented with the easier problems. Although the less able students were presented with the separation no regrouping problem types in Years 3 and 4, the level of accuracy continued to rise (to 86.6%). Subtractive strategies were predominantly used in the first 3 interviews, but by interview 6, additive strategies were also used. However, subtractive strategies were more accurate than additive strategies.

As with the separation problems, missing-addend problems were also attempted by fewer children in interview 6, as more students were attempting regrouping problems (Table 4, B). However, the accuracy level fell in this interview. Although additive strategies were dominant throughout, subtractive had become significant by interview 6, and were consistently more accurate.

Presentation of vertical and horizontal algorithms without regrouping ceased in interview 4, to allow for more difficult question types to be presented (reported in Cooper, Heirdsfield, & Irons, 1995a, 1995b, in press). Data shown in Table 4, C and D, indicate the preference and higher accuracy levels for subtractive strategies. However, additive strategies were present by interview 2 for vertical algorithms and by interview 3 for horizontal algorithms.

For separation word problems with regrouping, subtractive strategies were most popular; however, additive strategies were significant in interviews 2 and 3, and combination strategies were also quite popular in interviews 5 and 6 (Table 4, E). Additive strategies proved to be the most accurate of the strategies in interviews 2 (equal with subtractive), 3, and 5 (although the numbers were quite small). Combined strategies were most accurate in interview 6. However, none were particularly accurate, but this is consistent with the overall accuracy levels for subtraction with regrouping. Although percentages attempting the regrouping missing-addend question type increased over the interviews, few children (37.5%) were capable of attempting it even in interview 6 (Table 4, F). The dominant strategies throughout were additive; however, by interview 6, subtractive had also become significant. Subtractive strategies were more accurate in this interview, although the small numbers makes it hazardous to make such assumptions.

The percentage of students attempting the vertical algorithm increased over the interviews, although accuracy levels did not improve (Table 4, G). This fact could be accounted for by the dramatic increase in number, but not ability. Subtractive strategies remained dominant; however, a few children also chose additive strategies in interviews 5 and 6, although, with no success. The low accuracy levels for subtractive are consistent with the low accuracy levels for subtraction in general. The horizontal algorithm for regroup exercises was not presented beyond interview 3 (Table 4, H). As mentioned above, this was to allow for presentation of more difficult problems and exercises. Subtractive were the sole strategies employed for this question type.

Discussion and Conclusions

The children in this study exhibited both subtractive and additive strategies for all separation and missing-addend word problems with and without regrouping. This is consistent with the findings of Carpenter and Moser (1984). Further, the children used both subtractive and additive strategies for algorithmic exercises, with and without regrouping. This was also reported by Perry and Stacey (1994) in their study of much older students (years 8 to 12). However, in this present study, subtractive strategies were predominantly used by the younger children.

For word problems, the children initially used strategies that related to the semantic structure of the problem. Similar findings were reported by Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) and De Corte and Verschaffel (1985). However, in line with the findings of Carpenter (1984), this relationship was not as strong by Year 4 (i.e., subtractive for separation and additive for missing-addend had reduced). Unlike Carpenter and Moser (1984), this weakening of relationship moved both ways. Both the use of additive strategies for separation problems and the use of subtractive strategies for missing-addend problems increased, although subtractive for missing-addend increased more. There was not the emphasis on additive strategies that was present in the studies by Carpenter and Moser (1984), Fuson (1986a, 1986b), Fuson and Willis (1988), Secada (1982), and Steffe, Spikes, and Hirsten (1976). In fact, for problems with regrouping, children's strategy use showed a tendency to move towards subtractive. By Year 4, the use of subtractive strategies for missing-addend problems with regrouping was higher than for problems without regrouping while the use of additive strategies for separation problems with regroup was much lower.

For algorithmic exercises, both vertical and horizontal, subtractive strategies strongly predominated (91.7 to 100%). This was in line with the findings for young children and basic facts (Steinberg, 1985) and for older children (Perry & Stacey, 1994).

Unlike much of the literature, the children in this study did not prefer the additive strategies. Initially, they followed the semantic structure of the word problems, and preferred subtractive strategies for algorithmic exercises. There was some evidence that the subtractive strategies began to predominate as the children became older.

References

Boulton-Lewis, G.M. (1993). An assessment of the processing load of some strategies and representations for subtraction used by teachers and young children. *Journal of Mathematical Behavior*, *12*, 387-409.

Carpenter, T.P. (1986). Conceptual knowledge as a foundation for procedural knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Carpenter, T.P., Ansell, E., Franke, M.L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education, 24*, 428-441.

Carpenter, T.P., Franke, M.L., Jacobs, V.R., & Fennema, E. (1996, April). *Invention and understanding of multidigit addition and subtraction procedures: A* *longitudinal study*. Paper to be presented at the conference of the American Educational Research Association, New York.

Carpenter, T.P., & Moser, J.M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *The acquisition of mathematical concepts and processes*. New York: Academic Press.

Carpenter, T.P., & Moser, J.M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179-202.

Cooper, T.J., Heirdsfield, A., & Irons, C.J. (1995a). Years 2 and 3 children's strategies for mental addition and subtraction. In B. Atweh & S. Flavel (Eds.) *Proceedings of 18th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 195-202). Darwin: Mathematics Education Research Group of Australasia .

Cooper, T.J., Heirdsfield, A., & Irons, C.J. (1995b, November). *A comparison of years 2 and 3 children's mental strategies for algorithmic exercises and word problems*. Paper presented at the annual conference of the Australian Association of Research in Education, Hobart, Australia.

Cooper, T.J., Heirdsfield, A., & Irons, C.J. (in press). Years 2 and 3 children's mental strategies for addition and subtraction word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Research in early number learning: An Australasian perspective*. Sydney: Mathematics Education Research Group of Australasia .

De Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' solution strategies of elementary addition and subtraction word problems. *The Journal for Research in Mathematics Education, 18*(5), 363-81.

Fuson, K.C. (1986a). Teaching children to subtract by counting up. *Journal for Research in Mathematics Education, 3*, 172-189.

Fuson, K.C. (1986b). Roles of representation and verbalization in the teaching of multi-digit addition and subtraction. *European Journal of Psychology of Education, 1*(2), 35-56.

Fuson, K.C., & Willis, G.B. (1988). Subtracting by counting up: more evidence. *Journal for Research in Mathematics Education, 19*, 402-420.

Ginsburg, H. P., Kossan, N. E., Schwartz, R., & Swanson, D. (1983). Protocol methods in research on mathematical thinking. In H. P. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic.

Heirdsfield, A.M. (1996). *Mental computation, computational estimation, and number fact knowledge for addition and subtraction in year four children*. Unpublished master's thesis, Queensland University of Technology, Brisbane.

Perry, A.D., & Stacey, K. (1994). The use of taught and invented methods of subtraction. *Focus on Learning Problems in Mathematics, 16*(3), 12-22.

Secada, W.G. (1982, March). *The use of counting for subtraction*. Paper presented at the annual meeting of the American Educational Research Association, New York.

Sowder, J. (1988). Making sense of numbers in school mathematics. In G. Leinhardt, R. Putman, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics*. Hillsdale, NJ: Erlbaum.

Steffe, L.P., Spikes, W.C., & Hirsten, J.J. (1976). *Summary of quantitative comparisons and class inclusion as readiness variables for learning first-grade arithmetical content* (Working Paper No. 1). Athens: University of Georgia, Center for Research in the Learning and Teaching of Mathematics.

Steinberg, R. (1985). Instruction on derived facts strategies in addition and subtraction. *Journal for Research in Mathematics Education, 16*, 337-355.

Thornton, C.A. (1990). Solution strategies: Subtraction number facts. *Educational Studies in Mathematics, 21*, 241-263.